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Propagation of Kelvin waves along irregular coastlines in finite-difference models

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In this paper, we examine the behavior of internal Kelvin waves on an *f*-plane in finite-difference models using the Arakawa C-grid. The dependence of Kelvin wave phase speed on offshore grid resolution and propagation direction relative to the numerical grid is illustrated by numerical experiments for three different geometries: (1) Kelvin wave propagating along a straight coastline; (2) Kelvin wave propagating at a 45° angle to the numerical grid along a stairstep coastline with stairstep size equal to the grid spacing; (3) Kelvin wave propagating at a 45° angle to the numerical grid along a coarse resolution stairstep coastline with stairstep size greater than the grid spacing. It can be shown theoretically that the phase speed of a Kelvin wave propagating along a straight coastline on an Arakawa C-grid is equal to the analytical inviscid wave speed and is not dependent on offshore grid resolution. However, we found that finite-difference models considerably underestimate the Kelvin wave phase speed when the wave is propagating at an angle to the grid and the grid spacing is comparable with the Rossby deformation radius. In this case, the phase speed converges toward the correct value only as grid spacing decreases well below the Rossby radius. A grid spacing of one-fifth the Rossby radius was required to produce results for the stairstep boundary case comparable with the straight coast case. This effect does not appear to depend on the resolution of the coastline, but rather on the direction of wave propagation relative to the grid. This behavior is important for modeling internal Kelvin waves in realistic geometries where the Rossby radius is often comparable with the grid spacing, and the waves propagate along irregular coastlines.

Keywords: hydrodynamic modeling, finite difference, Kelvin wave.

1 INTRODUCTION

Kelvin waves are shore-trapped long gravity waves on a rotating plane in which geostrophic balance is maintained between the Coriolis force due to the longshore velocity and the offshore pressure gradient. The amplitude of the Kelvin wave decays exponentially in the offshore direction with an e-folding distance, or Rossby radius, of $r = c/f$, where c is the Kelvin wave phase speed and f is the Coriolis parameter. The phase speed can range from $10\ \mathrm{m\ s^{-1}}$ for barotropic tides in estuaries and seiches in large lakes to $0.1\ \mathrm{m\ s^{-1}}$ for internal waves in weakly stratified estuaries or lakes. Therefore, at mid-latitudes the Rossby radius can vary from 100 km for barotropic modes to 1 km for baroclinic modes. The latter value is often comparable with or smaller than the grid spacing in numerical hydrodynamic models, so grid resolution becomes a particular concern when modeling baroclinic coastal waves and upwelling.

As shown by Hsieh *et al.*[1] and Davey *et al.*,[2] the propagation speed of Kelvin waves in a numerical model can be affected by the finite difference scheme, and by lateral and vertical viscosity. They tested two of the most popular finite difference grids used in ocean modeling, the Arakawa B-grid and the Arakawa C-grid. They found that as the grid spacing increases (resolution worsens), the Kelvin wave speed increases dramatically in the Arakawa B-grid, but stays constant at the long gravity wave speed in the Arakawa C-grid in the inviscid case. Increasing lateral or vertical viscosity decreases the wave velocity for both grids. The effect of finite differencing on the properties of coastal Kelvin waves was also investigated by Henry[3] and Wajsowicz and Gill.[4] All of these results were obtained for Kelvin waves propagating along a straight coastline.

Beletsky *et al.*[5] tested two three-dimensional numerical

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coastal ocean models for internal Kelvin wave response with more complex geometries, including a circular basin with a flat bottom, a circular basin with a sloping bottom, and actual Lake Michigan geometry. They found that both of the numerical models systematically underestimated Kelvin wave speed in the circular lake and Lake Michigan cases, even though one of the models used the Arakawa-C grid (the second model used the Arakawa-A grid) and lateral viscosity was low. Their results showed some improvement as grid spacing was reduced to less than half the Rossby radius, but were still much less than the ideal value. These results led us to investigate other possible causes for the low Kelvin wave speed.

In this paper, we examine the effect of irregular geometry and propagation direction on Kelvin wave propagation in numerical finite difference models. Our results apply mainly to internal Kelvin waves where the numerical grid spacing is comparable with the Rossby radius. We first eliminate the effects of viscosity, non-linearity, bottom topography, and three-dimensionality by considering a reduced gravity model. We then compare the results from the reduced gravity model with a full three-dimensional model. To isolate the effect of shoreline geometry, three basin geometries are considered. One is a 100 km square basin with straight shorelines parallel to the grid coordinates. The second is a similarly sized basin, but oriented at a 45° angle to the grid coordinates so that the shoreline is represented by a stairstep boundary in the numerical model. The third geometry is similar to the second, but with a coarse resolution stairstep shoreline. We will discuss the behavior of Kelvin wave propagation in these three geometries and compare results from the reduced gravity and three-dimensional models.

2 MODELS

2.1 Reduced gravity model

The dynamical equations for a two-layer stratified fluid can be separated into two components, one for the barotropic response and one for the baroclinic response. The form of the equations in each set is identical, but the interpretation of the coefficients and independent variables is different for the two modes. The linearized inviscid equations can be written as

$$u_t - fv = -gHh_x - \alpha\rho^{-1}\tau^x \tag{1}$$

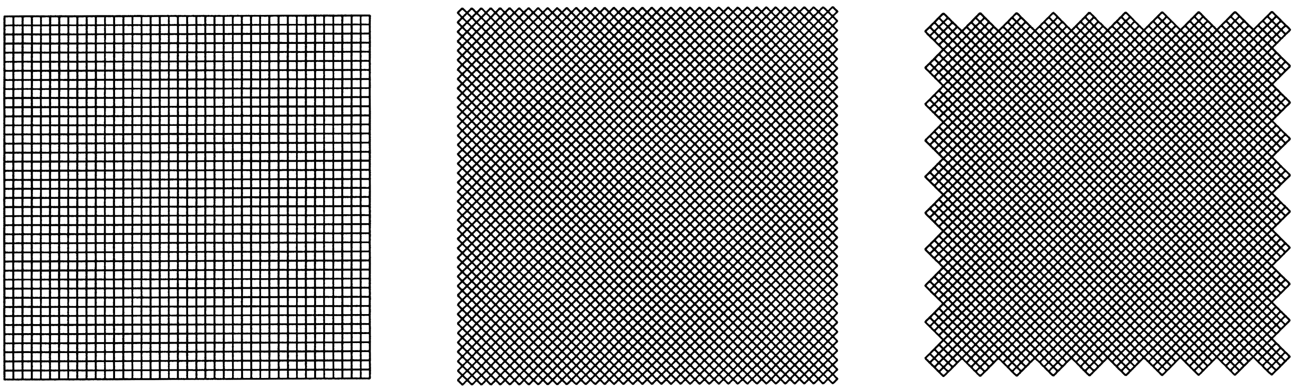
$$v_t + fu = -gHh_y - \alpha\rho^{-1}\tau^y \tag{2}$$

$$h_t + u_x + v_y = 0 \tag{3}$$

For the baroclinic mode, u and v are the lower layer transports, h is the interface displacement, g is the reduced gravity (gravity multiplied by the relative density difference between the layers), H is the equivalent depth (the product of the layer depths divided by their sum), α is the ratio of the lower layer depth to the total depth, and $\rho^{-1}\tau^x$ and $\rho^{-1}\tau^y$ are the wind forcing terms. The Coriolis parameter f is set to $10^{-4}\ \mathrm{s}^{-1}$, a typical value for mid-latitudes. Eqns (1)–(3) are solved in finite difference form on an Arakawa C-grid. We use a finite difference scheme introduced by Sielecki.[6] Time differencing is forward for eqn (3) and backward for eqns (1) and (2), except for the Coriolis term in eqn (2) which is treated by forward differencing. This scheme is commonly used in storm surge modeling and has the advantage that computer storage is required for only one time level of each of the three prognostic variables.

2.2 Three-dimensional model

For many coastal hydrodynamic applications, a full three-dimensional model is required. The Princeton Ocean Model (Blumberg and Mellor[7]) is a non-linear, primitive equation, finite difference model that solves the three-dimensional, hydrostatic, Boussinesq, Navier–Stokes equations. It has been used extensively for coastal applications and was used in the study of Kelvin waves in Lake Michigan by



(a) (b) (c)

Fig. 1. Examples of model grids: (a) 2.5 km straight shoreline grid, (b) 1.768 km stairstep shoreline grid, and (c) 1.768 km coarse stairstep shoreline grid. All basins are 100 km square and 100 m deep.

Table 1. Model geometries and Kelvin wave propagation speed for reduced gravity model

Case number	Cell size (km)	Effective $\Delta x/r$	c/c_0
Straight coastline			
1	10.000	3.367	0.92
2	5.000	1.684	0.99
3	2.500	0.842	1.01
4	1.250	0.421	1.02
5	0.625	0.210	1.02
Stairstep coastline (grid rotated 45°)			
6	7.071	3.367	0.64
7	3.536	1.684	0.77
8	1.768	0.842	0.88
9	0.884	0.421	0.94
10	0.442	0.210	0.98
Coarse stairstep coastline (grid rotated 45°)			
11	3.536	1.684	0.77
12	1.768	0.842	0.84
13	0.884	0.421	0.86
14	0.442	0.210	0.86

Beletsky *et al*.[5] The equations are written in flux form and finite differencing is done on an Arakawa C-grid. The finite difference scheme is second-order and centered in space and time (leapfrog). The vertical coordinate is normalized by depth (sigma coordinate), but this feature is not significant for the purposes of this paper. The model employs a mode splitting technique that uses different time steps for barotropic and baroclinic modes. The model includes a turbulence closure scheme for calculating vertical mixing coefficients as prognostic variables. Horizontal diffusion is calculated with a Smagorinsky eddy diffusion parameterization (with a multiplier of 0.1).

2.3 Model geometries and boundary conditions

In order to isolate the effect of irregular coastline on Kelvin wave propagation, three model geometries are considered. The first is a 100 km square basin aligned with the grid coordinates. The second is the best approximation to a 100 km square basin that one can make when the basin is oriented at a 45° angle to the grid coordinates. The third geometry is similar to the second, but with a coarse resolution of the stairstep shoreline. Fig. 1 shows the 2.5 km version of the straight coastline grid and the 1.768 km version of the two stairstep grids. Note that the distance between successive grid points along a line in the offshore direction is the same (2.5 km) in all three grids. We also use 10, 2.5, 1.25, and 0.625 km grids for the straight coastline grid and 7.071, 1.768, 0.884, and 0.442 km grid spacing for the stairstep grids. As shown in Table 1, the ''effective $\Delta x/r$'' is the ratio of the offshore grid spacing to the Rossby radius for each grid.

All basins are 100 m deep. Both the reduced gravity and the three-dimensional model are initialized to represent a two-layer system with a relative density discontinuity of $(\rho_2 - \rho_1)/\rho_2 = 10^{-3}$ at 10 m depth. The inviscid long internal gravity wave speed for this vertical profile is $0.297\ \mathrm{m\ s^{-1}}$ and the Rossby radius is 2.97 km.

The initial condition for each case is a state of rest. A Kelvin wave is generated in the basin by allowing a uniform wind stress of 0.1 dyne $\mathrm{cm^{-2}}$ from the north to act on the basin for 1 day. In each case, the Kelvin wave is allowed to propagate for 14 days after the wind ceases. At $0.297\ \mathrm{m\ s^{-1}}$, the Kelvin wave can propagate 359 km in 14 days.

3 RESULTS

3.1 Reduced gravity model

The reduced gravity model was applied to the three basin geometries at the five different grid resolutions shown in Table 1. Note that the lowest grid resolution for the coarse stairstep coastline case 6 is $\Delta x/r = 1.684$. The $\Delta x/r = 3.367$ coarse stairstep coastline would be the same grid as the stairstep coastline for this resolution (case 11). In each case, a contour plot of the interface displacement was created for each hour of the 15 day simulation. The contour plots were assembled into a computer animation of the Kelvin wave propagation. Samples of single frames from the animations are shown in Fig. 2. The full animations are available on the journal's homepage at **http://www.elsevier.nl/locate/advwatres** or **http://www.elsevier.com/locate/advwatres** (see ''Special section on Visualization''). As shown in Fig. 2(a), at the time the wind stops blowing (1 day) there is upwelling along the coast to the left of the wind direction and downwelling along the coast to the right. After the wind stops, the upwelling–downwelling pattern progresses counterclockwise around the basin. The wave speed is determined by estimating the total distance the Kelvin wave has traveled in 14 days. For case 12, illustrated in Fig. 2, this distance is approximately 300 km. Table 1 and Fig. 3 compare the ratios of the measured Kelvin wave speeds with the idealized wave speeds for all 14 cases.

3.2 Three-dimensional model

The three-dimensional model was applied to the two different basin geometries at the effective $\Delta x/r = 0.842$ grid resolution only. The vertical grid levels were set at 0, 4, 7, 9, 10, 11, 13, 16, 20, 30, 50, and 100 m below the surface to provide higher vertical resolution in the thermocline region. Two different values of background vertical viscosity were used in the three-dimensional model, $1 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$ and $2 \times 10^{-5}\ \mathrm{m^2\ s^{-1}}$—corresponding respectively to molecular viscosity and to a more typical value used in coastal circulation models. The Kelvin wave phase speed was calculated

Table 2. Kelvin wave speed in three-dimensional model ($\Delta x/r = 0.842$)

Background vertical viscosity ($\mathrm{m^2\ s^{-1}}$)	Straight coastline c/c_0	Stairstep coastline c/c_0
1×10^{-6}	0.96	0.80
2×10^{-5}	0.82	0.63

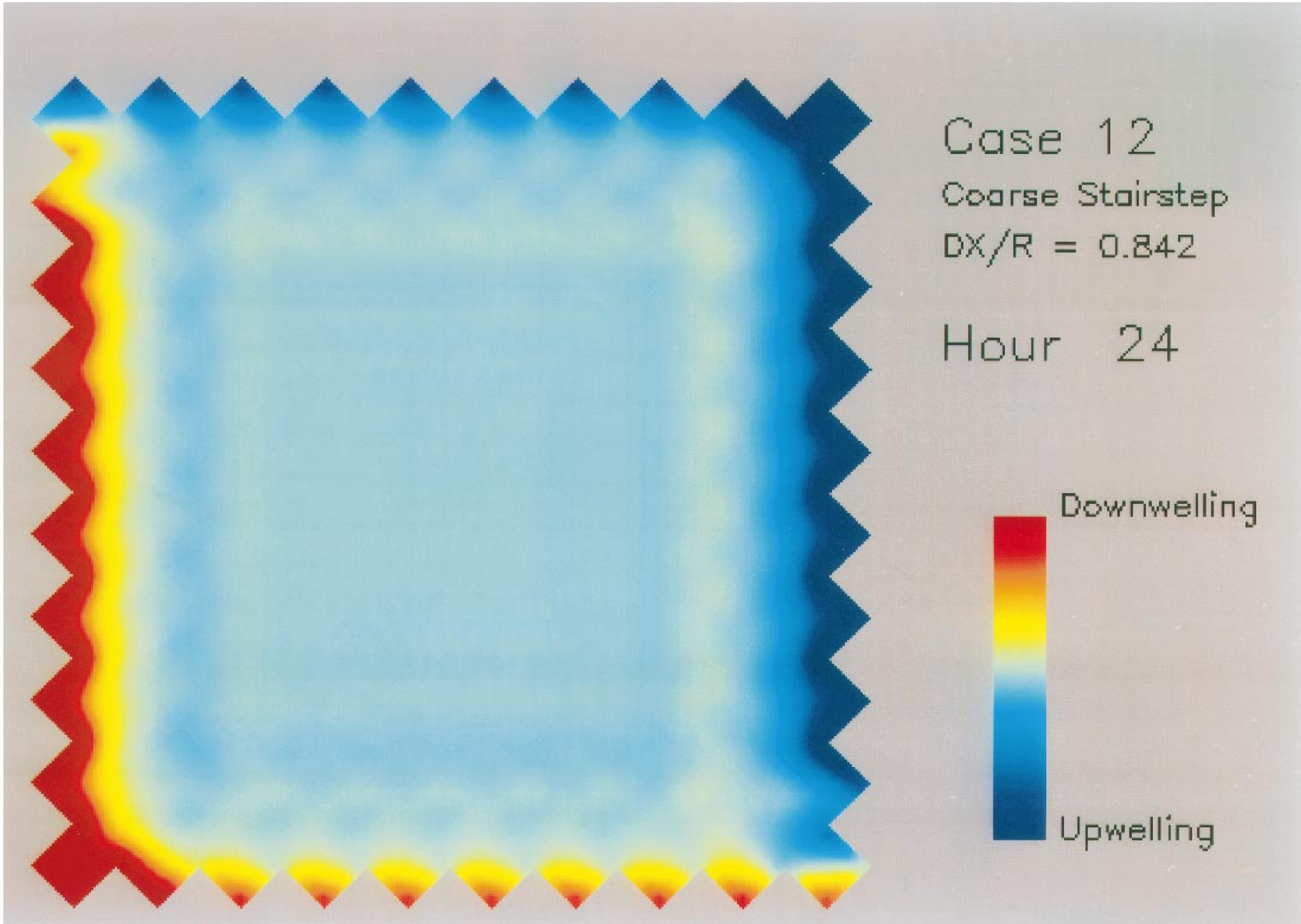


Fig. 2. Contour plot of the interface displacement (a) after 1 day of northerly wind stress, and (b) after 14 days of free wave propagation for case 12, the coarse stairstep coastline with $\Delta x/r = 0.842$.

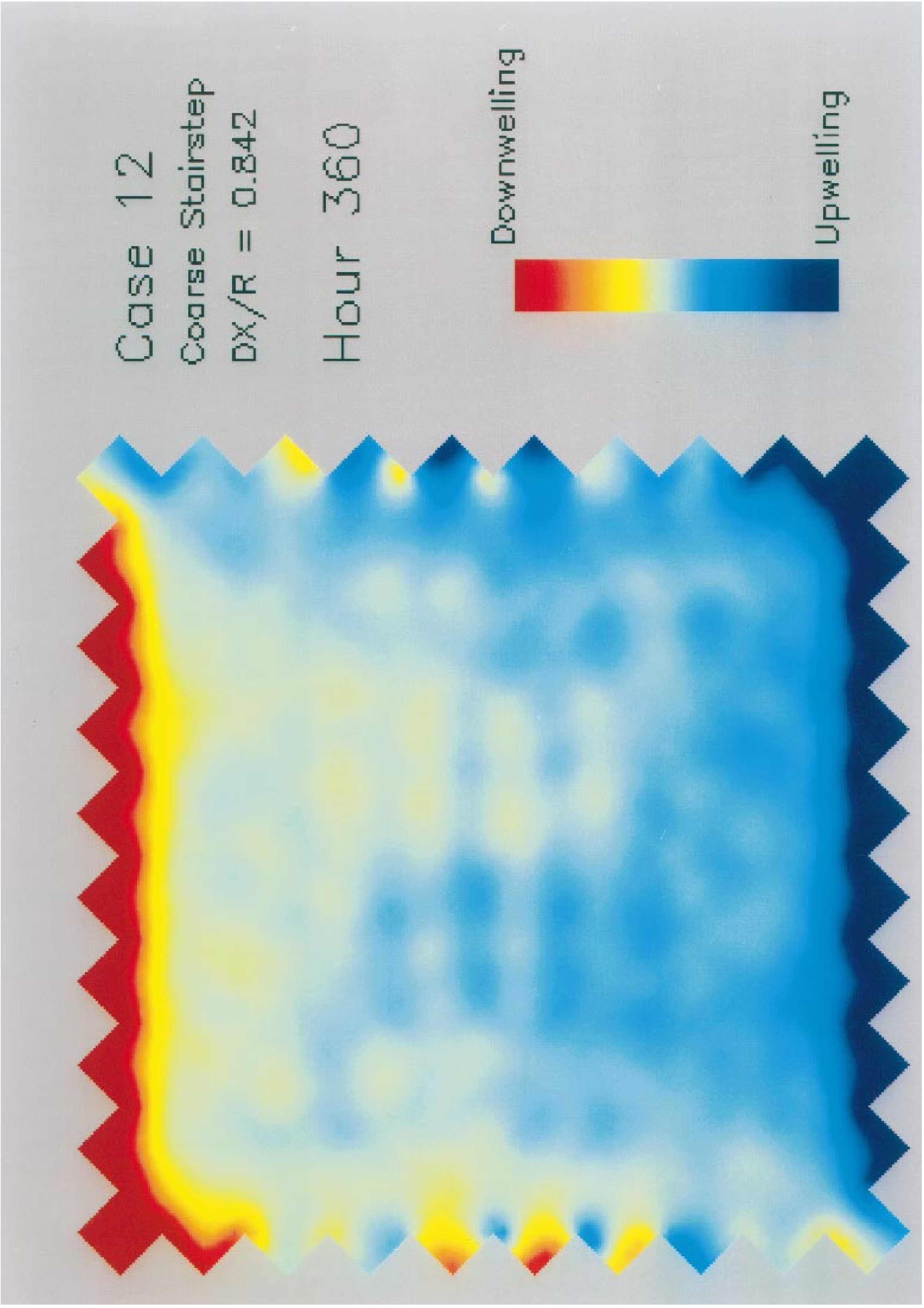


Fig. 2. (*continued*).

by the same technique as was used for the reduced gravity model, except we used a contour plot of temperature at 10.5 m instead of interface displacement. The results from the four tests are shown in Table 2.

4 DISCUSSION

4.1 Reduced gravity model

As shown in Table 1 and Fig. 3, the reduced gravity model generates a Kelvin wave with a speed very close to the ideal value for the straight coastline cases (1–5) where the Kelvin wave propagation is aligned with the grid orientation, in agreement with the theoretical result of Hsieh *et al.*[1] The estimated propagation speed slightly exceeds the theoretical speed in some of the high resolution cases (3–5). This is probably due to the inherently approximate method we use for estimating the propagation speed in the model results. An examination of the computer animations shows that the propagation speed of the Kelvin wave is not constant, and the estimated speed depends on the exact position of the wave front on day 15. In contrast, the Kelvin wave speed is considerably lower in the lower resolution stairstep coastline cases (6–8) where the Kelvin wave propagation is at an angle to the grid orientation. The speed is only 53% of the ideal case when the grid spacing is 3.4r. It appears to converge gradually to the ideal value as the grid spacing decreases (cases 9 and 10). For a grid spacing of 0.2r the speed improves to 98% of the ideal value (case 10). The cases with a coarse resolution stairstep boundary (11–14) where the shoreline configuration is a 7.071 km stairstep like in case 6, show that the reduction in Kelvin wave speed is due to the fact that the wave is propagating at an angle to the grid, and not because of the resolution of the shoreline. In cases 13 and 14 there is no improvement in propagation speed; this is because grid resolution becomes less than the Rossby radius and the Kelvin wave sees a longer shoreline than 400 km. This is clear in the computer animations for these cases. Note that we obtained nearly identical results for all these cases with a version of the reduced gravity model using a strictly centered time stepping scheme, eliminating the possibility of wave speed distortion from the Sielecki[6] scheme.

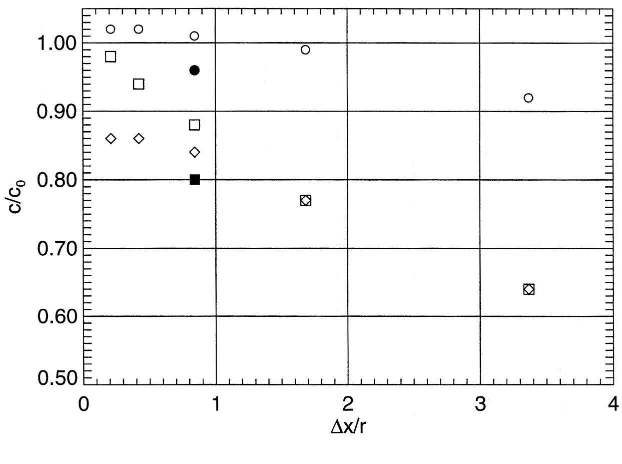


Fig. 3. Ratio of Kelvin wave speed to the ideal value ($0.297\ \mathrm{m\,s^{-1}}$) in finite-difference model test cases. Open circles are for the reduced gravity model, straight coastline cases. Open squares are for the reduced gravity model, stairstep coastline cases. Open diamonds are for the reduced gravity model, coarse stairstep coastline cases. The filled circle is the result from the low vertical viscosity three-dimensional model with a straight coastline. The filled square is the result from the low vertical viscosity three-dimensional model with a stairstep coastline. The theoretical value of c/c_0 for a Kelvin wave along a straight coast is 1.0 (Hsieh *et al.*[1]).

4.2 Three-dimensional model

Our main purpose in using the three-dimensional model was to estimate possible additional effects of non-linearity and viscosity on Kelvin wave speed. We found that the Princeton Ocean model produced almost identical results to the reduced gravity model, but only if the background vertical viscosity was reduced to a value near molecular viscosity. For example, with a background vertical viscosity typical for lake applications ($2 \times 10^{-5}\ \mathrm{m^2\,s^{-1}}$) the Kelvin wave speed is reduced by 15–20% over the inviscid case for 0.8r grid resolution. This result is consistent with the findings of Hsieh *et al.*[1] and Davey *et al.*[2] that increasing vertical viscosity decreases the Kelvin wave speed. This also may explain to some extent the low Kelvin wave speed reported by Beletsky *et al.*,[5] where the emphasis was more on the effect of horizontal viscosity. Based on the results shown in Table 2, the circular lake geometry was probably an even more important factor in their findings.

5 CONCLUSIONS

Our main conclusion is that propagation at an angle to the numerical grid in some finite difference models can considerably reduce the phase speed of Kelvin waves when the grid size is comparable with or greater than the Rossby radius. In particular, for the Arakawa C-grid, adequate representation of the Kelvin wave phase speed requires a grid spacing less than about 20% of the Rossby radius. Previous studies (Hsieh *et al.*,[1] Wajsowicz and Gill,[4] and Henry[3]) have only dealt with Kelvin waves propagating along a straight coast aligned with the numerical grid. The results presented here for Kelvin waves propagating at a 45° angle to the numerical grid along a stairstep coast apply to three-dimensional as well as two-dimensional reduced gravity models. From a practical point of view, until an analytical solution to this problem is found, it may be worthwhile testing coastal finite difference models with computational grids other than the Arakawa-C grid using geometries like the stairstep and coarse startstep grids used here. The relatively high resolution required for accurate representation of Kelvin wave phase speed is often not feasible for many coastal applications, so if Kelvin wave dynamics play a critical role in a particular application, other strategies such as boundary-fitted coordinates, nested grids, or finite elements may be necessary.

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